Lecture 5
Excitatory-Inhibitory and stochastic Networks

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Excitatory-Inhibitory Network

starting from the equation from the output rate

$$\tau_r \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \mathbf{F}(\mathbf{h} + \mathbf{M} \cdot \mathbf{v}) \tag{1}$$

Dale's law: neuron have either excitatory or inhibitory effects on all of their postsynaptic targets

$M_{aa'}$ strength of synapses from a' to a

- neuron a' excitatory $\rightarrow M_{aa'} > 0 \; \forall a$
- neuron a' inhibitory $\rightarrow M_{aa'} < 0 \; \forall a$

describe these neurons separately

$$\tau_E \frac{d\mathbf{v}_E}{dt} = -\mathbf{v}_E + \mathbf{F}_E(\mathbf{h}_E + \mathbf{M}_{EE} \cdot \mathbf{v}_E + \mathbf{M}_{EI}\mathbf{v}_I)$$

$$\tau_I \frac{d\mathbf{v}_I}{dt} = -\mathbf{v}_I + \mathbf{F}_I(\mathbf{h}_I + \mathbf{M}_{IE} \cdot \mathbf{v}_E + \mathbf{M}_{II}\mathbf{v}_I)$$

note, that a symmetric $\mathbf{M}$ violates Dale's law

Illustration of the dynamics – a simple model

all excitatory neurons are described by a *single* firing rate v_E, and all inhibitory neurons by another *single* firing rate v_I

$F(\cdot)$ threshold linear function

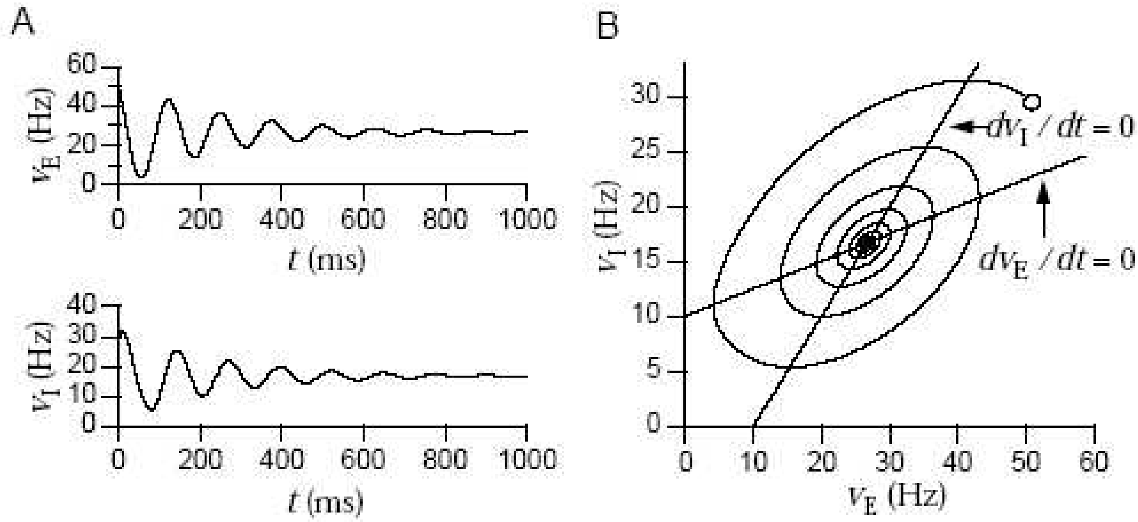
$\Rightarrow$

$$\tau_E \frac{dv_E}{dt} = -v_E + [M_{EE} \cdot v_E + M_{EI} v_I - \gamma_E]_+$$

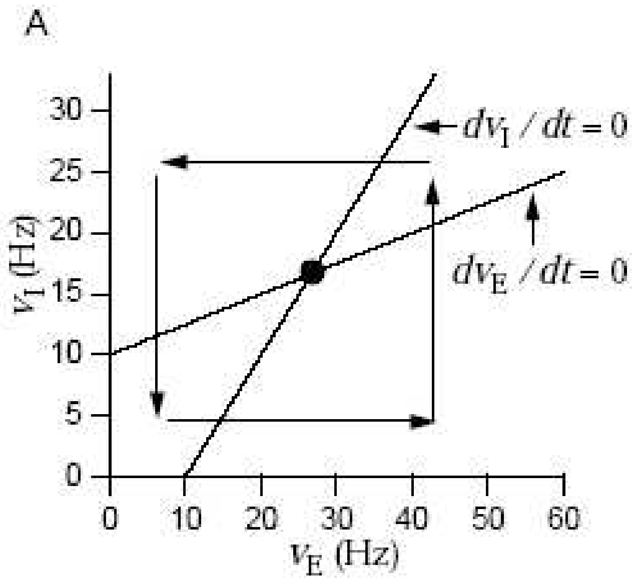
$$\tau_I \frac{dv_I}{dt} = -v_I + [M_{IE} \cdot v_E + M_{II} v_I - \gamma_I]_+ \tag{2}$$

We set $M_{EE} = 1.25$, $M_{iE} = 1$, $M_{II} = 0$, $M_{EI} = -1$, $\gamma_E = -10$ Hz, $\gamma_I = 10$ Hz , $\tau_E = 10$ ms; and we vary τ_I

dynamical behavior – fixed points



Activity of the excitatory-inhibitory firing-rate model when the fixed point is stable. A) The excitatory and inhibitory firing rates settle to the fixed point over time. B) The phase-plane trajectory is a counter-clockwise spiral collapsing to the fixed point. The open circle marks the initial values $v_E(0)$ and $v_I(0)$. For this example, $\tau_I = 30$ ms.



Nullclines, flow directions, and fixed points

stability Analysis

fixed point is

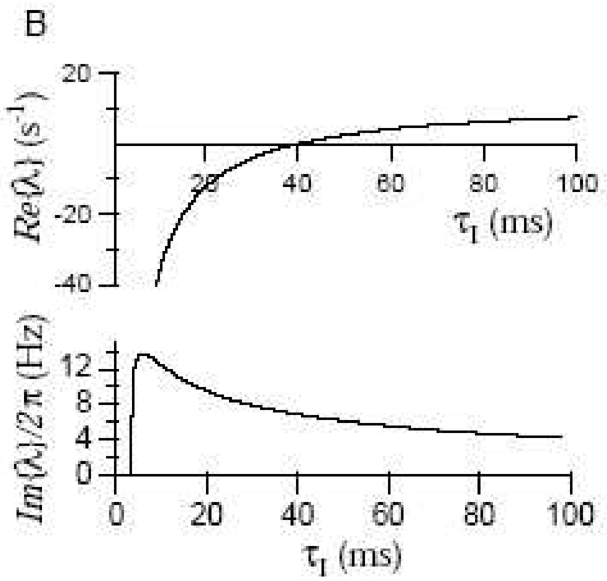
- **stable** → initial values of v_E and v_I near this point will be drawn toward it over time
- **unstable** → nearby configurations are pushed away from the fixed point

stability of the fixed point is determined by the real parts of the eigenvalues of the matrix

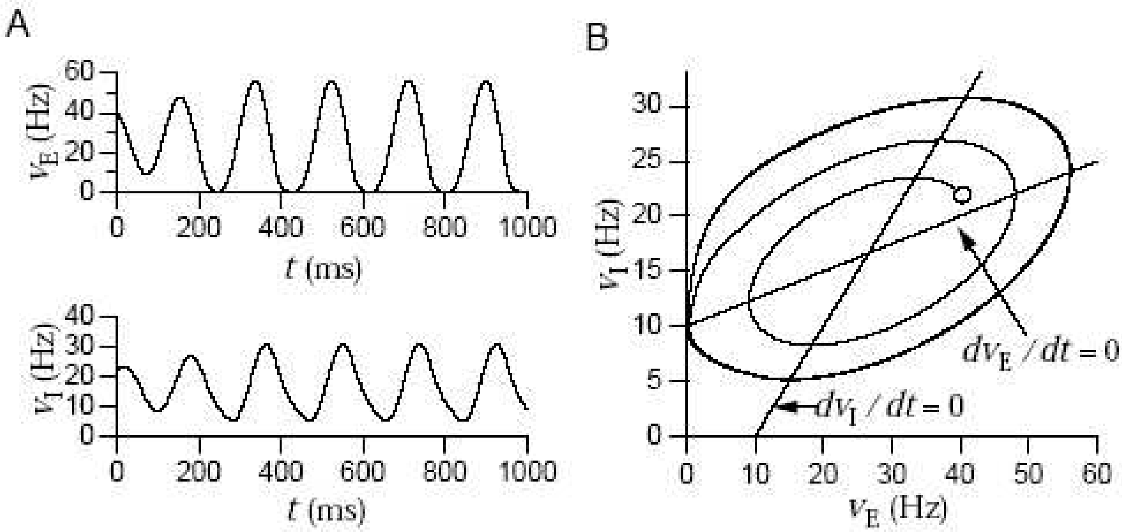
$$\begin{pmatrix} (M_{EE}-1)/\tau_E & M_{EI}/\tau_E \\ M_{IE}/\tau_I & (M_{II}-1)/\tau_I \end{pmatrix}.$$

eigenvalues are

$$\lambda = \frac{1}{2}\left(\frac{M_{\mathrm{EE}}-1}{\tau_{\mathrm{E}}} + \frac{M_{\mathrm{II}}-1}{\tau_{\mathrm{I}}} \pm \sqrt{\left(\frac{M_{\mathrm{EE}}-1}{\tau_{\mathrm{E}}} - \frac{M_{\mathrm{II}}-1}{\tau_{\mathrm{I}}}\right)^2 + \frac{4M_{\mathrm{EI}}M_{\mathrm{IE}}}{\tau_{\mathrm{E}}\tau_{\mathrm{I}}}}\right)$$



real and imaginary part of the eigenvalue determining the stability of the fixed point
⇒ fixed point is stable for $\tau_I < 40$ ms and unstable for larger values of τ_I



Activity of the excitatory-inhibitory firing-rate model when the fixed point is unstable. A) The excitatory and inhibitory firing rates settle into periodic oscillations. B) The phase-plane trajectory is a counter-clockwise spiral that joins the limit cycle, which is the closed orbit. The open circle marks the initial values $v_E(0)$ and $v_I(0)$. For this example, $\tau_I = 50$ ms.

bifurcation: transition from a stable fixed points to a limit cycle

Exercise 1

Write a matlab program to analyze the dynamical behavior of the system of differential equations (2). Plot also a phase-plane trajectory.

Stochastic Networks

consider the total input current of unit a with symmetric $\mathbf{M}$ (and see Eq. (1))

$$I_a(t) = h_a(t) + \sum_{a'=1}^{N_v} M_{aa'} v_{a'}(t) \tag{3}$$

Boltzmann machine: (stochastic neurons):
If single unit a is selected, then update is done as follows:
v_a is set to 1 with *probability*:

$$P[v_a(t+\Delta t) = 1] = F(I_a(t)), \quad \text{with } F(I_a) = \frac{1}{1+\exp(-2\beta I_a)} \tag{4}$$

and to 0 otherwise; $\beta = 1/T$ with "temperature" T.
Using update rule (4) $\mathbf{v}$ does not converge to a fixed point, but can be described by a probability distribution

$$P[\mathbf{v}] \propto \exp(-\beta E(\mathbf{v})), \quad E(\mathbf{v}) = -\mathbf{h}\cdot\mathbf{v} - \frac{1}{2}\mathbf{v}\cdot\mathbf{M}\cdot\mathbf{v} \tag{5}$$

associated with an energy function $E(\mathbf{v})$.
Note: $T = 0$ in Eq. (4) $\Rightarrow$ $F(\cdot)$ is threshold linear function and $\mathbf{v}$ evolves according to Eq. (1).

statistical physics – Ising model

The idea of Eq. (4) can be derived with methods of statistical physics.

Gibbs sampling – canonical ensemble:
system with energy $E(s)$ in a heat reservoir with *temperature* T is in the thermodynamical equilibrium in state s with probability (Boltzmann distribution)

$$P(s) = \frac{\exp[-\beta E(s)]}{Z}, \tag{6}$$

with partition function $Z \equiv \sum_s \exp[-\beta E(s)]$ and $\beta = 1/(kT)$.

System with two states:
example: single Ising spin in a magnetic field h:
$s = \pm 1$, $E(s) = -sh$

$$\Rightarrow P(s = \pm 1) = \frac{1}{1+\exp(\mp 2\beta h)} \tag{7}$$

Hopfield model:
set $h_a(t) \equiv 0$ in Eq. (3) $\Rightarrow$ Hopfield model (with stochastic neurons)
if $T = 0$ we have the deterministic Hopfield model (N recurrent neurons with threshold linear function

$$S_i = \operatorname{sgn}\left[\sum_{j=1}^{N} M_{ij} S_j\right], \quad S_i = \pm 1 \tag{8}$$

Ising model
physical analogy to the Hopfield model

$$H = -\frac{1}{2}\sum_{ij} M_{ij} S_i S_j \tag{9}$$

with $S_i = \pm 1$
H (Hamiltonian) is an *energy function* for the Hopfield model, meaning that, if $H \rightarrow H'$ according to the dynamic of the Hopfield model, than $H' \leq H$
Exercise 2: Show that for the deterministic Hopfield model (with $M_{ii} \geq 0$)

Ising model with

- $T = 0$: equivalent to deterministic Hopfield model
- $T > 0$: equivalent to Hopfield model with stochastic neurons

mean field approximation

is an general approximation in statistical physics
example: Ising model

for Hopfield model with stochastic neurons follows in mean field approximation

$$\langle S_i \rangle = \tanh(\beta \sum_j M_{ij} \langle S_j \rangle) \tag{10}$$

Exercise 3: Do the mean field approximation for various temperatures using matlab for

$$M_{ij} = \begin{pmatrix} 0 & 0.5 & 0.3 \\ 0.5 & 0 & 0.4 \\ 0.3 & 0.4 & 0 \end{pmatrix}.$$